INTRODUCTION TO CATEGORY THEORY

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Contents

Introduction

These notes were written as part of a project *Pares Ordenados* from January 2023 to May 2023 under the supervision of Juan Omar Gomez. The goal of the present work is to give a survey of the very basics of Category Theory, which nowadays is a very important tool in mathematics allowing us to compare different structures. We emphasize in the *Category of Modules* and give an immediate application of this theory by doing the *Grothendieck Group*. In chapters 1 and 2 we follow the reference [Lei14], [Alu21] and [Awo10]. For chapter 3, the reference was [Bly18] and [RR09]. For chapter 4 [Ros12].

Acknowledgments. I thank Juan Omar Gomez because of his constant support and for his direction.

1. Categories

We start by defining what a Category is and give plenty examples to illustrate the concept. After this we see that there is a category in which objects are categories and morphisms between objects are (what is known as) functors. We end this section with natural transformations and equivalence of categories.

Definition 1.1. A category $\mathcal{C}$ consists of the following data.

- A collection of *objects* $\mathrm{Ob}(\mathcal{C})$.
- For every $A, B \in \mathrm{Ob}(\mathcal{C})$, a collection of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
- For every $A, B, C \in \mathrm{Ob}(\mathcal{C})$, there exists a map
$$\circ\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C).$$
such that the following two properties hold
- For every $A \in \mathrm{Obj}(\mathcal{C})$, there exists a distinguish morphism $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ called the *identity* in A and it is such that for every $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$,
$$1_B \circ f = f = f \circ 1_A.$$

Date: May 5, 2023.

- For every $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B,C)$, $h \in \mathrm{Hom}(C,D)$ we have,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

From now on, a morphism from A to B will be sometimes denoted by $f\colon A \to B$.

Example 1.2. The following is a list of some well-known categories.

1. Set is the category whose objects are sets and the morphisms are functions. The composition of morphisms is defined by the usual function composition.
2. Let k be a field. Thus Vec_k is the category whose objects are k-vector spaces and the morphisms are linear transformations. The composition between two linear transformations is given by the composition of the morphisms as functions of sets.
3. Grp is the category whose objects are groups and morphisms are group homomorphisms. Similarly one can define the category of rings Ring and the category of topological spaces Top.
4. Let $\leq$ be a transitive and reflexive relation on the set S. We want to define a category with this information. Define the objects of the category $\mathcal{C}$ as the elements in S and $\mathrm{Hom}_{\mathcal{C}}(a,b) = *$ consists of a single element if $a \leq b$ and empty otherwise. The composition of morphisms is given by the transitive property. If the relation is given by $a \leq b$ if and only if $a = b$, then there are no non-trivial morphisms. Such categories are called **discrete**.
5. Let X be a topological space. Consider the relation
$$y \leq x \text{ if and only if } x \in U \Rightarrow y \in U \text{ for every open U.}$$
If X is T_0, then this relation is a partial order (that is, a reflexive, anti-symmetric and transitive relation). Therefore the poset[1] $(X, \leq)$ determines a category (see Example 4).
6. Ord is the category whose objects are pairs $(S, \leq)$ with $\leq$ a reflexive and transitive relation on the set S and morphisms are order-preserving functions. The composition in this case is just the usual composition of functions between sets. In a similar fashion, we define the category of posets which is denoted by Poset.
7. Let R be a ring. We define the category $\mathcal{C}$ as follows: $\mathrm{Obj}(\mathcal{C}) = \mathbb{Z}^+$ and $\mathrm{Hom}_{\mathcal{C}}(n,m) = \mathbb{M}_{n \times m}(R)$ and the composition between morphisms is given by matrix multiplication. The category $\mathcal{C}$ is usually denoted by Mat_R and it is known as the category of matrices with entries in the ring R.
8. Let $(\mathrm{G}, \cdot)$ be a group. The category BG has only one object (say $\star$) and the morpisms correpond to the elements of the group. The composition of morphisms is given by the multiplication of the group. Explicitly, let $g \in \mathrm{Hom}(\star,\star)$ and $h \in \mathrm{Hom}(\star,\star)$, then $h \circ g := h \cdot g$.
9. Let $\mathcal{C}$ be a category and let A be an object in $\mathcal{C}$. Consider the category $\mathcal{C}_A$ whose objects are given by the morphisms $Z \to A$ in $\mathcal{C}$, and morphisms between two objects $Z_1 \to A$, $Z_2 \to A$ are given by a morphisms $\varphi \in \mathrm{Hom}_{\mathcal{C}}(Z_1, Z_2)$ such that the following triangle is commutative.
$$\begin{tikzcd} Z_1 \arrow[rr, dashed, "\varphi"] \arrow[dr] & & Z_2 \arrow[dl] \\ & A & \end{tikzcd}$$

[1]Poset is short for partially ordered set.

The composition is (as expected) defined by means of doing it in $\mathcal{C}$, i.e, if we have two morphisms ψ, φ

$$\begin{tikzcd}
Z_1 \arrow[rr, "\psi"] \arrow[dr] & & Z_2 \arrow[dl] & Z_2 \arrow[rr, "\varphi"] \arrow[dr] & & Z_3 \arrow[dl] \\
& A & & & A &
\end{tikzcd}$$

then, their composition is $\varphi \circ \psi$

$$\begin{tikzcd}
Z_1 \arrow[rr, "\varphi\circ\psi"] \arrow[dr] & & Z_3 \arrow[dl] \\
& A &
\end{tikzcd}$$

10. Now, given two objects A, B in $\mathcal{C}$, define $\mathcal{C}_{A,B}$ as the category in which objects are diagrams

$$\begin{tikzcd}
& Z \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

and a morphism between two objects

$$\begin{tikzcd}
& Z_1 \arrow[dl] \arrow[dr] & & & Z_2 \arrow[dl] \arrow[dr] & \\
A & & B & A & & B
\end{tikzcd}$$

is a morphisms $\varphi\colon Z_1 \to Z_2$ in $\mathcal{C}$ which makes the following diagram commutative.

$$\begin{tikzcd}
& Z1 \arrow[d, dashed, "\varphi"] \arrow[ddl] \arrow[ddr] & \\
& Z2 \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

The composition is induced by the composition in $\mathcal{C}$.

11. Let $\mathcal{C}$ be a category and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(A, C)$, $\beta \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Let $\mathcal{C}_{\alpha,\beta}$ denote the category whose objects are commutative diagrams

$$\begin{tikzcd}
& A \arrow[dr, "\alpha"] & \\
Z \arrow[ur] \arrow[dr] & & C \\
& B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

and a morphism between two objects

$$\begin{tikzcd}
 & A \arrow[dr, "\alpha"] & \\
Z_1 \arrow[ur] \arrow[dr] & & C \\
 & B \arrow[ur, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dr, "\alpha"] & \\
Z_2 \arrow[ur] \arrow[dr] & & C \\
 & B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

is an element $\varphi \in \mathrm{Hom}_{\mathcal{C}}(Z_1, Z_2)$ such that the following diagram commutes.

$$\begin{tikzcd}
 & & A \arrow[dr, "\alpha"] & \\
Z_1 \arrow[r, dashed, "\varphi"] \arrow[urr, bend left] \arrow[drr, bend right] & Z_2 \arrow[ur] \arrow[dr] & & C \\
 & & B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

12. We finish this list of examples with a very geometric one. The following are called **n** categories. They have n objects, the required identity morphisms are omitted and the other morphisms are shown in the diagrams below.
   - **n=1**

$$\bigstar$$

   - **n=2**

$$\begin{tikzcd}
\bigstar \arrow[r] & \blacklozenge
\end{tikzcd}$$

   - **n=3**

$$\begin{tikzcd}
\bigstar \arrow[r] \arrow[dr] & \blacklozenge \arrow[d] \\
 & \clubsuit
\end{tikzcd}$$

   - **n=4**

$$\begin{tikzcd}
\bigstar \arrow[r] \arrow[d] \arrow[dr] & \blacklozenge \arrow[d] \\
\spadesuit \arrow[r] \arrow[ur] & \clubsuit
\end{tikzcd}$$

   - **n=5**

$$\begin{tikzcd}
 & \blacklozenge \arrow[dr] \arrow[ddr] & \\
\spadesuit \arrow[ur] \arrow[rr] \arrow[drr] & & \bigstar \arrow[d] \\
\clubsuit \arrow[u] \arrow[uur] \arrow[urr] \arrow[rr] & & \blacksquare
\end{tikzcd}$$

2. Functors and natural transformations

In this section we focus on functors, these corresponds to the morphisms in the category of categories as we will see.

Convention: From now on we restrict our treatment to categories with the following property. For any pair of objects a, b, the morphisms from a to b determine a set. This categories are known as *locally small categories*.

Definition 2.1. Let $\mathcal{C}, \mathcal{D}$ be categories. A *(covariant) functor* $F\colon \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is a map sending objects of $\mathcal{C}$ to objects of $\mathcal{D}$ and morphisms of $\mathcal{C}$ to morphisms of $\mathcal{D}$ preserving the structure of the category, i.e., it preserves domains, codomains, composition and identities. Explicitly:

- $F(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FB$
- $F(A \xrightarrow{f} B \xrightarrow{g} C) = FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$
- $F(A \xrightarrow{1_A} A) = FA \xrightarrow{1_{FA}} FA$

If the first condition is changed for $F(A \xrightarrow{f} B) = FB \xrightarrow{Ff} FA$, then we say that F is a *contravariant functor*.

Example 2.2. The following are some of the most straightforward examples of functors.

1. Let $\mathcal{C}$ be a category. and consider the functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which maps every object and every morphism to itself.
2. The endofunctor $\mathcal{P} : \textsc{Set} \to \textsc{Set}$ which assigns every set its power set and maps every function $f\colon X \to Y$ to the set function $f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by $f(A \subseteq X) = f(A) \subseteq Y$. In this example, by taking inverse image instead of direct image we obtain a contravariant functor.
3. For categories with structure (groups, rings, modules, posets, etc.) one can consider the functor which "forgets" the additional structure. For example, $F_1\colon \textsc{Poset} \to \textsc{Set}$, $F_2\colon \textsc{Ring} \to \textsc{Set}$, $F_3\colon \textsc{Top} \to \textsc{Set}$. There is also a functor that forgets only a part of the structure, i.e., $F\colon {}_R\textsc{Mod} \to \textsc{Ab}$.
4. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $b \in \mathrm{Obj}(\mathcal{D})$. Define the constant functor $b^{\mathcal{C}}\colon \mathcal{C} \to \mathcal{D}$ by $b^{\mathcal{C}}(X) = b$ and $b^{\mathcal{C}}(f) = 1_b$ for every object X and morphism f in $\mathcal{C}$, respectively.
5. Let $\mathcal{C}$ be a category and $A \in \mathrm{Obj}(\mathcal{C})$. Then we have a functor $Hom_{\mathcal{C}}(A, \_)\colon \mathcal{C} \to \textsc{Set}$ which maps an object B in $\mathcal{C}$ to $Hom_{\mathcal{C}}(A, B)$ and maps a morphism $B \xrightarrow{f} C$ to $Hom_{\mathcal{C}}(A, B) \xrightarrow{f^*} Hom_{\mathcal{C}}(A, C)$ which is given by $f^*(g) = f \circ g$. In particular, these functors are of great interest for this project when $\mathcal{C} =_R \textsc{Mod}$.
6. The functor $\mathrm{GL}_n(-)\colon \textsc{Ring} \to \textsc{Grp}$ maps a ring R to its associated general linear group $\mathrm{GL}_n(R)$ and maps every ring homomorphism to the group homomorphism obtained by applying the ring homomorphism component-wise to the matrices.
7. The functor $(-)^{\times}\colon \textsc{Ring} \to \textsc{Group}$ taking every ring and sending it to its group of units.
8. The functor $(-)^{\mathrm{op}}\colon \mathcal{C} \to \mathcal{C}$ sending every object to itself and every morphism to the same morphism but with the arrow pointing in the opposite direction.
9. The functor $(-)^*\colon \textsc{Vec}_k \to \textsc{Vec}_k$ assigns to each vector space its dual vector space and each linear transformation to its transpose, is contravariant.

10. The functor Spec: $\textsc{CRing} \to \textsc{Top}$ sends every commutative ring R to $Spec\, R$ the set of all prime ideals of R with the Zariski Topology and every ring morphism $R_1 \xrightarrow{f} R_2$ to $\text{Spec}R_2 \xrightarrow{f^*} \text{Spec}R_1$ defined by $f^*(p) = f^{-1}(p)$.
11. In some categories one can think of free objects such as free groups, rings, modules,... so it makes sense to talk about a Free functor as the following examples illustrate
    i. Given a set S, there is the free group $F(S)$ associated to S and every $S \xrightarrow{f} S'$ set function induces a group homomorphism $F(S) \xrightarrow{f^*} F(S')$ sending $x = a_1^{r_1} a_2^{r_2} \dots a_n^{r_n} \in F(S)$ to $f^*(x) = f(a_1)^{r_1} f(a_2)^{r_2} \dots f(a_n)^{r_n}$.
    ii. Similarly, construct the commutative free ring $F(S)$ defined by polynomials in $\mathbb{Z}$ with variables x_s $(s \in S)$. For example, if $S = \{x, y\}$ then $F(S) = \mathbb{Z}[x, y]$.
    iii. For every non-empty set and commutative ring R one can always construct the free $R-$module on S which is essentially copies of R indexed by S, i.e., $\bigoplus_{s\in S} R_s$.
12. Define $\pi_1 \colon \textsc{Top}_* \to \textsc{Group}$ as the functor which assigns to every pointed space (X, x_0) its fundamental group $\pi_1(X, X_0)$ and to every continuous function $(X, x_0) \xrightarrow{f} (Y, y_0)$ the group homomorphism $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0)$ sending every loop g based in x_0 to the loop $f \circ g$ based in y_0.
13. Let BG be the monoid G viewed as a category (see Example 1.2.8). A functor $F \colon \text{BG} \to \textsc{Set}$

    corresponds to the following information. A set S which is the image of the only object of BG and for every $g \in G$ a function $F_g \colon S \to S$ say $F_g(s) = g \cdot s$ and is such that $(g'g) \cdot s = g' \cdot (g \cdot s)$ and $1 \cdot s = s$. Then a functor from BG to $\textsc{Set}$ corresponds to a set S together with a left action of G, this is called a left G–set. Similarly, a contravariant functor between these categories is a right G–set.
14. Let $\mathcal{A}, \mathcal{B}$ be posets viewed as categories. A functor from $\mathcal{A}$ to $\mathcal{B}$ is an order-preserving function.
15. Let $n \in \mathbb{Z}^+$. The map $F \colon \textsc{CRing} \to \textsc{Mon}$ sending a commutative ring R to $\mathbb{M}_n(R)$ and every ring morphism $f \colon R \to S$ to $f^* \colon \mathbb{M}_n(R) \to \mathbb{M}_n(S)$ such that $f^*(a_{ij}) = (f(a))_{ij}$, i.e, we change every entry of the matrix for its image by f. This prescription defines a functor.
16. Let $\mathcal{A}$ be a category. A presheaf in $\mathcal{A}$ is a contravariant functor from $\mathcal{A}$ to $\textsc{Set}$. Note that this is equivalent to specify a functor from the opposite category, i.e, a presheaf is a functor from $\mathcal{A}^{op}$ to $\textsc{Set}$.

Now, consider functors $F \colon \mathcal{A} \to \mathcal{B}$, $G \colon \mathcal{B} \to \mathcal{C}$. We can define the composition $H = G \circ F \colon \mathcal{A} \to \mathcal{C}$ by $HA = (G \circ F)(A)$ and $Hf = (G \circ F)(f)$ for every object A and morphism f in $\mathcal{A}$, one can check that this defines a functor. From this observation it is clear that we can form a category whose objects are categories and morphisms are functors.

Definition 2.3. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there exist functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G = 1_{\mathcal{D}}$ y $G \circ F = 1_{\mathcal{C}}$.

Example 2.4. The category of abelian groups is isomorphic to the category of $\mathbb{Z}$–modules.

Definition 2.5. Let $G, F\colon \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\eta\colon F \Rightarrow G$ is a collection of maps $\{\eta_A\colon F(A) \to G(A)\}_{A\in\mathcal{C}}$ such that for every morphism $f\colon A \to B$ the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"] & G(B) \end{tikzcd}$$

A natural transformation between functors F, G will be sometimes written as follows

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Define $1_F\colon F \Rightarrow F$, where each $(1_F)_A$ is the identity morphism 1_A. On the other hand, for two natural transformations $\eta\colon F \Rightarrow G$ and $\eta'\colon G \Rightarrow H$, define the composition as $(\eta' \circ \eta)_A := \eta'_A \circ \eta_A$ so that we have a natural transformation $\eta' \circ \eta\colon F \Rightarrow H$. The definition above is known as the category functor from $\mathcal{C}$ to $\mathcal{D}$ denoted by $\mathcal{D}^{\mathcal{C}}$ or $[\mathcal{C}, \mathcal{D}]$. Now we give some examples to illustrate what a natural transformation is.

Example 2.6. Let $\mathcal{A}$ be a discrete category whose objects are positive integers, then a functor $F\colon \mathcal{A} \to \mathcal{B}$ is essentially objects $F_1, F_2, \ldots, F_n, \ldots$ so a natural transformation $\alpha\colon F \to G$ of two such functors is a collection of morphisms $\alpha_i\colon F_i \to G_i$.

Example 2.7. Let n be a fixed natural number. We already know that $\mathbb{M}_n\colon \mathrm{CR{\small ING}} \to \mathrm{M{\small ON}}$ define a functor. We can view every ring $(R, +, \cdot)$ as a monoid $(R, \cdot)$ and so we also have a functor $U\colon \mathrm{CR{\small ING}} \to \mathrm{M{\small ON}}$. We want to prove that $(\det_R\colon \mathbb{M}_n(R) \to U(R))_{R\in\mathrm{CR{\small ING}}}$ is a natural transformation, i.e,

$$\begin{tikzcd} \mathrm{CR{\small ING}} \arrow[r, bend left, "\mathbb{M}_n"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\Downarrow\det"] & \mathrm{M{\small ON}} \end{tikzcd}$$

For this, let $f\colon R \to S$ be a ring morphism, then,

$$\begin{tikzcd} \mathbb{M}_n(R) \arrow[r, "\det_R"] \arrow[d] & U(R) \arrow[d] \\ \mathbb{M}_n(S) \arrow[r, "\det_S"] & U(S) \end{tikzcd}$$

is a commutative diagram since $f(\det_r(a_{ij})) = \det_S(f(a_{ij}))$.

We end this section with two important and useful definitions.

Definition 2.8. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and F, G two functors from $\mathcal{C}$ to $\mathcal{D}$. We say F and G are natural isomorphic if they are isomorphic in the category $[\mathcal{C}, \mathcal{D}]$.

Definition 2.9. An equivalence of categories $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms

$$\eta\colon 1_{\mathcal{C}} \to G \circ F, \qquad \varepsilon\colon F \circ G \to 1_{\mathcal{D}}.$$

3. Categories of Modules

In this section we present some basics of the category of modules which led to define the Grothendieck group of a ring. In the following let R be a commutative ring with unity.

Definition 3.1. Let $(M_i)_{i\in I}$ a family of R–modules. By a product of this family we mean a pair $(P,(f_i))_{i\in I}$ where P is an R–module and $f_i\colon P \to M_i$ are R–morphisms, such that for any other pair $(M,(g_i)_{i\in I})$ there exists a unique R–morphism h from M to P making the following diagram commutes

$$\begin{tikzcd} & M \arrow[dl, dashed, "h"'] \arrow[d, "g_i"] \\ P \arrow[r, "f_i"] & M_i \end{tikzcd}$$

If such product exists then it is unique up to isomorphism.

Proposition 3.2. *If $(M_i)_{i\in I}$ is a family of R–modules then the cartesian product of this family $\prod\limits_{i\in I} M_i$ (viewed as an R–module with the operation and action component wise) is a product of this family.*

Similarly to the definition of a product, we define the dual notion, namely a coproduct.

Definition 3.3. Let $(M_i)_{i\in I}$ be a family of R–modules. By a coproduct of this family we mean a pair $(C,(f_i))_{i\in I}$ where C is an R–module and $f_i\colon M_i \to C$ are R–morphisms, such that for any other pair $(M,(g_i)_{i\in I})$ there exists a unique R–morphism h from C to M making the following diagram commute

$$\begin{tikzcd} M_i \arrow[r, "g_i"] \arrow[d, "f_i"'] & M \\ C \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

If such coproduct exists, then it's unique up to isomorphism.

Proposition 3.4. *If $(M_i)_{i\in I}$ is a family of R–modules then the direct sum $\bigoplus\limits_{i\in I} M_i$ is a coproduct of this family.*

Definition 3.5. If M, N are right and left R–modules respectively then by a tensor product of these we mean a pair (T,f) where T is a $\mathbb{Z}$–module and $f: M \times N \to T$ is a R-biadditive map, such that for any other pair (P,g) with g also R-biadditive, there exists a unique $\mathbb{Z}$–morphism $h\colon P \to T$ making the following diagram commute

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[d, "f"'] & P \\ T \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

If such a tensor product exists then it is unique up to isomorphism, and it is denoted by $M \otimes_R N$.

Let M, N be right and left R–modules respectively. Consider F the free $\mathbb{Z}$–module with basis $M \times N$ and let S be the subgroup generated by all elements of the following type,

- $(a, b+b') - (a,b) - (a,b')$
- $(a+a', b) - (a,b) - (a',b)$
- $(ar, b) - (a, rb)$

for every $(a,b) \in M \times N$ and $r \in R$. Define $M \otimes_R N = F/S$, $h\colon M \times N \to M \otimes_R N$ by $h(a,b) = a \otimes_R b$ where $a \otimes_R b = (a,b) + S$.

Proposition 3.6. *The $\mathbb{Z}$–module $M \otimes_R N$ defined above is a tensor product of M, N.*

Proposition 3.7. *Let M, N_i be left R–modules and S a right R–module. Then, the tensor product we have the following two properties:*

- $R \otimes_R M \cong M$
- $S \otimes_R \bigoplus_{i \in I} N_i \cong \bigoplus_{i \in I} (S \otimes_R N_i)$.

Theorem 3.8. *Let R, S be commutative rings with 1, M an $SR-$bimodule and N a left $R-$module. Then $M \otimes_R N$ is an S–module.*

Definition 3.9. Let M be an R–module and $\emptyset \neq X \subseteq M$. Similar to vector spaces, we say X is a basis for M if X is a linearly independent which generates M.

Definition 3.10. Let S be a non-empty set. A free R–module F over S is a pair (F, f) where F is an R–module and $f\colon S \to F$ is a set function, satisfying that for every R–module M and every set function $g\colon S \to M$ there exists an R–morphism $h\colon F \to M$ such that the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "f"'] & M \\ F \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Moreover, if (F, f) is free, then f is injective and $\mathrm{im} f$ is a basis for F.

Proposition 3.11. *An R–module is free if and only if it has a basis.*

Corollary 3.12. *Every free R–module is isomorphic to a direct sum of copies of R.*

More can be said, if S is a non-empty set and (F, f) is a free R–module over S, then F has $\mathrm{im} f$ as a basis. Since f is injective, we have that S is in bijection with $\mathrm{im} f$. Thus we can consider S as a basis for F which is isomorphic to R–module $\bigoplus_{s \in S} R_s$.

Proposition 3.13. *Every R–module is a quotient of a free module.*

Definition 3.14. A sequence of R–modules is a diagram

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots$$

A sequence of R–modules is called exact at M_i if $\mathrm{im}\, f_{i-1} = \ker f_i$. It is said to be exact whenever it is exact for every M_i in the sequence.

Definition 3.15. An exact sequence $N \to P \to 0$ is *right split* if there exists $\alpha\colon P \to N$ making the following diagram commutative

$$\begin{tikzcd} & P \arrow[dl, dashed, "\alpha"'] \arrow[d, "id_P"] & \\ N \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

Similarly, an exact sequence $0 \to M \to N$ is *left split* if there exists $\beta\colon N \to M$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "id_M"'] & N \arrow[dl, dashed, "\beta"] \\ & M & \end{tikzcd}$$

A short exact sequence $0 \to M \to N \to P \to 0$ is said to split if it is left split and right split.

Proposition 3.16. *If $0 \to M \to N \to P \to 0$ splits, then $N \cong M \oplus P$.*

Definition 3.17. Let M be an R–module and N an R-submodule of M. We say N is a retract of M if there is an R–morphism $r\colon M \to N$ such that the following diagram commutes

$$\begin{tikzcd} & M \arrow[dr, "r"] & \\ N \arrow[ur, hook, "i"] \arrow[rr, "id_N"'] & & N \end{tikzcd}$$

where i is the inclusion. Which means, N is retract if the inclusion splits on the left.

Definition 3.18. An R–module P is called projective if every diagram with exact row

$$\begin{tikzcd} & P \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

can be extended to a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$

Lemma 3.19. *Let P be a projective module and N be a retract of P. Then N is projective.*

Lemma 3.20. *If $M \subseteq P$ and M is a direct summand of P, then M is a retract of P.*

Lemma 3.21. *Every free module is projective.*

Theorem 3.22. *Let P be an $R-$module. The following properties are equivalent:*

(i) *P is projective.*
(ii) $\mathrm{Hom}_R(P, \_)$ *is exact.*
(iii) *Every exact sequence $M \to P \to 0$ splits.*
(iv) *P is a direct summand of a free module.*

Definition 3.23. Let $\mathcal{C}$ be a category. A lifting problem is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

This problem has a solution if there is $d\colon B \to X$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "d"] & Y \end{tikzcd}$$

Fixing f, g we say f has the left lifting property with respect to g if every lifting problem with vertical arrows f and g has a solution. In this case we also say g has the right lifting property with respect to f and we write $f \nearrow g$.

Remark 3.24. If M is a colection of morphisms of a category $\mathcal{C}$, we write $f \nearrow M$ meaning $f \nearrow m$ for every $m \in M$. For example, if $\mathcal{C} = \textsc{Set}$ then mono $\nearrow$ epi.

We can rephrase the definition of a projective module making use of lifting properties as follows

Definition 3.25. An R–module P is projective if $(0 \to P) \nearrow$ epi as the diagram illustrates

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M \arrow[d, two heads, "g"] \\ P \arrow[r] \arrow[ur, dashed] & N. \end{tikzcd}$$

4. The Grothendieck group

In this section we study what is known as the Grothendieck construction which is used later to introduce the Grothendieck group of a ring. For this purpose, we recall that one can build the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$ and the idea is that $\mathbb{Z} := \mathbb{N} \times \mathbb{N}/\sim$ and $\mathbb{N} \subseteq \mathbb{Z}$, where the relation is defined as $(m_1, n_1) \sim (m_2, n_2)$ if and only if $m_1 + n_2 = m_2 + n_1$. Similarly for the rational numbers we have $\mathbb{Q} := \mathbb{Z} \times \mathbb{Z}/\sim$ by applying the same method but using the multiplication for the relation. In general, the question is whether for every abelian monoid $(M, +, 0)$ we can form a group G by doing the construction explained before such that $M \subseteq G$; the answer is positive and corresponds precisely to the *Grothendieck construction.*

Theorem 4.1. *Let $(M, +, 0)$ be an abelian monoid. Then there exists an abelian group $K(M)$ and a morphism of monoids $i_M\colon M \to K(M)$ such that for every other abelian group G and monoid morphism $f\colon M \to G$ there exists a unique group morphism $\hat{f}\colon K(M) \to G$ making the following diagram commute*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i_M"'] & G \\ K(M) \arrow[ur, dashed, "\hat{f}"'] & \end{tikzcd}$$

Proof. Consider the relation in $M \times M$ given by $(m_1, n_1) \sim (m_2, n_2)$ if and only if there is $u \in M$ such that $m_1 + n_2 + u = m_2 + n_1 + u$. We claim that this is an equivalence relation:

- **Reflexive** : Since $m + n + 0 = m + n + 0$ then $(m, n) \sim (m, n)$.
- **Symmetric** : Let $(m_1, n_1) \sim (m_2, n_2)$ then $m_1 + n_2 + u = m_2 + n_1 + u$ which is the same as $m_2 + n_1 + u = m_1 + n_2 + u$, hence $(m_2, n_2) \sim (m_1, n_1)$.
- **Transitive** : Let $(m_1, n_1) \sim (m_2, n_2) \sim (m_3, n_3)$, then we have the two equations $m_1 + n_2 + u = m_2 + n_1 + u$ and $m_2 + n_3 + v = m_3 + n_2 + v$. Adding those and letting $w = m_2 + n_2 + u + v$ we have $m_1 + n_3 + w = m_3 + n_1 + w$, hence $(m_1, n_1) \sim (m_3, n_3)$.

Now, let $K(M) = M \times M/ \sim$, the elements of this set will be written $[m, n]$ instead of $[(m, n)]$. Define in $K(M)$ an operation by

$$[m_1, n_1] + [m_2, n_2] = [m_1 + m_2, n_1 + n_2]$$

it is well defined since given $(m_1, n_1) \sim (m_1', n_1')$ and $(m_2, n_2) \sim (m_2', n_2')$ we have the equations $m_1 + n_1' + u = m_1' + n_1 + u$ and $m_2 + n_2' + v = m_2' + n_2 + v$, letting $w = u + v$ we get the equation $(m_1 + m_2) + (n_1' + n_2') + w = (m_1' + m_2') + (n_1 + n_2) + w$ by adding the two above. Hence $[m_1 + m_2, n_1 + n_2] = [m_1' + m_2', n_1' + n_2']$. Moreover, this operation satisfies the following properties:

- **Associative/Abelian** : It follows since the sum of classes is the class of the sum in the monoid which is associative and commutative.
- **Identity** : Let $[m, n], [k, k] \in K(M)$ then $[m, n] + [k, k] = [m + k, n + k] = [m, n]$ and similarly $[k, k] + [m, n] = [k + m, n + k] = [m, n]$. Then there is an identity element in $K(M)$ given by $[k, k]$ for every $k \in M$.
- **Inverses** : Let $[m, n] \in K(M)$ then $[m, n] + [n, m] = [m + n, m + n]$ implies $[n, m]$ is the additive inverse of $[m, n]$.

We conclude that $K(M)$ is an abelian group with this operation. Now, if $i_M : M \to K(M)$ is defined by $i_M(m) = [m, 0]$, then $i_M(0) = [0, 0]$ and $i_M(m + n) = [m + n, 0] = [m, 0] + [n, 0] = i_M(m) + i_M(n)$. In particular, i_M is a morphism of monoids.

Finally, if $f : M \to G$ is a monoid morphism and G is a group, then we need $\hat{f} : K(M) \to G$ group morphism such that $\hat{f} \circ i_M = f$. Define $\hat{f}([m, n]) = f(m) - f(n)$. If $(m_1, n_1) \sim (m_2, n_2)$, then we have an equation $m_1 + n_2 + u = m_2 + n_1 + u$, and by applying f, we obtain $f(m_1) + f(n_2) + f(u) = f(m_2) + f(n_1) + f(u)$ which is equal to $f(m_1) - f(n_1) = f(m_2) - f(n_2)$. Therefore $\hat{f}([m_1, n_1]) = \hat{f}([m_2, n_2])$. Now we shall prove that $\hat{f}$ is indeed a group morphism. This follows from

$$\begin{aligned}
\hat{f}([m_1, n_1] + [m_2, n_2]) &= \hat{f}([m_1 + m_2, n_1 + n_2]) \\
&= f(m_1 + m_2) - f(n_1 + n_2) \\
&= f(m_1) + f(m_2) - f(n_1) - f(n_2) \\
&= (f(m_1) - f(n_1)) + (f(m_2) - f(n_2)) \\
&= \hat{f}([m_1, n_1]) + \hat{f}([m_2, n_2])
\end{aligned}$$

□

Remark 4.2. In the construction presented above we did not assume that M has the cancellative property. In fact, one has the property that i_M is inyective if and only if M is cancellative.

Definition 4.3. Let R be a commutative ring. We define the Grothendieck group of the ring R as $K_0(R) = K((Proj\, R, \oplus))$ where $Proj\, R$ is the set of finitely generated projective left R–modules.

Now, we shall prove that with this definition, K_0 is a functor from the category of commutative rings CRing to the category of abelian groups Ab. For the sake of generality, we first prove that the Grothendieck construction of a monoid is a functorial. Thus $K_0 = K \circ Proj$ is a functor, since it is a composition of functors.

$$\begin{tikzcd} \textsc{Ring} \arrow[r, "Proj"] \arrow[dr, "K_0"'] & \textsc{AbMon} \arrow[d, "K"] \\ & \textsc{Ab} \end{tikzcd}$$

Consider the functor K which assigns to every abelian monoid M its Grothendieck construction $K(M)$ and to every morphism $f\colon M \to N$ the morphism $\overline{i_N \circ f}\colon K(M) \to K(N)$ given by $\overline{i_n \circ f}([m,n]) = [f(m), f(n)]$. Then, if $M_1 \xrightarrow{g} M_2 \xrightarrow{f} M_3$ we have to verify the equality $\overline{i_{M_3} \circ f} \circ \overline{i_{M_2} \circ g} = \overline{i_{M_3} \circ (f \circ g)}$,

- $\overline{i_{M_3} \circ (f \circ g)}([m,n]) = [f(g(m)), f(g(n))]$.
- $(\overline{i_{M_3} \circ f} \circ \overline{i_{M_2} \circ g})([m,n]) = \overline{i_{M_3} \circ f}([g(m), g(n)]) = [f(g(m), f(g(n))]$.

Moreover, if $M \xrightarrow{1_M} M$ then $\overline{i_M \circ 1_M}([m,n]) = [1_M(m), 1_M(n)] = [m,n] = 1_{K(M)}([m,n])$.

Now, define the functor $Proj$ which sends every commutative ring R to the monoid $(Proj\, R, \oplus)$ and every ring morphism $f\colon R \to S$ to the monoid morphism $f^*\colon Proj\, R \to Proj\, S$ given by the prescription $f^*(P) = S \otimes_R P$. Let's see that in fact this tensor product is in $Proj\, S$. Since P is projective then $P \oplus Q \cong R^n$ implies $(S \otimes P) \oplus (S \otimes Q) = S \otimes (P \oplus Q) \cong S \otimes_R R^n \cong S^n$, we conclude $S \otimes P$ is a direct summand of a free module then projective. The functor properties are shown below

- Let $R \xrightarrow{g} S \xrightarrow{f} T$ be ring morphisms, then $(f \circ g)^*(M) = T \otimes_R M \cong T \otimes_S S \otimes_R M = f^*(S \otimes_R M) = (f^* \circ g^*)(M)$.
- $(1_R)^*(M) = R \otimes_R M \cong M = 1_{Proj\, R}(M)$.
- Let $f\colon R \to S$ then $f^*(0) = 0$ and $f^*(M \oplus N) = S \otimes_R (M \oplus N) = (S \otimes_R M) \oplus (S \otimes N) = f^*(M) \oplus f^*(N)$. This means, f^* is in fact a monoid morphism.

We finish this document with some examples of the Grothendieck group of certain rings.

Example 4.4. If $R = F$ is a field then every finitely generated projective F–module is simply a finite dimensional vector space and the dimension is well-defined. Moreover, they are classified by its dimension. It follows that $Proj\, F \cong \mathbb{N}$ as monoids. Since K_0 is a functor, it preserves isomorphisms, then $K_0(F) \cong K_0(\mathbb{N}) = \mathbb{Z}$.

Example 4.5. Let R be a PID. Let M be a finitely generated projective R–module. Since M is projective then it is a direct summand of a free module, so M is embedded in R^n for some $n \in \mathbb{Z}^+$. We want to prove M is free, that is $M \cong R^k$ for some $k \in \mathbb{Z}^+$. By induction on n. If $n = 1$,

then $M \cong R$. Suppose that the result holds for every integer less than n. Let $\pi\colon R^m \to R$ denote the projection to the last component. Since M can be regarded as a subset of R^n, then we have the following two cases:

- $\pi(M) = 0$. Then $M \subseteq \ker\pi \leq R^{n-1}$. By the inductive hypothesis we conclude that $M \cong R^k$ for some $k \leq n-1 < n$.
- $\pi(M) \neq 0$. In this case we can consider $\pi(M)$ as an ideal of R. Since we are in a PID, it follows that $\pi(M)$ is principal, and hence projective. Consider the following exact sequence where i_M and i are the natural inclusions,

$$0 \to \ker(\pi \circ i_M) \xrightarrow{i} M \xrightarrow{\pi \circ i_M} \pi(M) \to 0$$

it splits since $\pi(M)$ is projective, so $M = \pi(M) \oplus ker(\pi \circ i_M)$, we know the second direct summand is embedded into R^{n-1} so by the inductive hypothesis it is isomorphic to some R^k with $k \leq n-1$. Hence $M \cong R^k \oplus \pi(M) \cong R^k \oplus R \cong R^{k+1}$ with $k+1 \leq n$.

Therefore $M \cong R^k$ for some k. Suppose that $M \cong R^k \cong R^m$. Let $F = \mathrm{Frac}(R)$. Then $F \otimes_R M \cong F \otimes_R R^k \cong F^k$. On the other hand, we have that $F \otimes_R M \cong F \otimes_R R^m \cong F^m$. Since vector spaces are determined, up to isomorphism, by their dimension, we deduce that $k = m$. It follows that k is unique. We conclude $K_0(R) \cong \mathbb{Z}$

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